

AN EULERIAN-LAGRANGIAN FORM FOR THE EULER EQUATIONS IN SOBOLEV SPACES

BENJAMIN C. POOLEY AND JAMES C. ROBINSON

ABSTRACT. In 2000 Constantin showed that the incompressible Euler equations can be written in an “Eulerian-Lagrangian” form which involves the back-to-labels map (the inverse of the trajectory map for each fixed time). In the same paper a local existence result is proved in certain Hölder spaces $C^{1,\mu}$.

We review the Eulerian-Lagrangian formulation of the equations and prove that if $n \geq 2$ and $s > \frac{n}{2} + 1$ then for initial data in H^s , a unique local-in-time solution exists on the n -torus that is continuous into H^s . These solutions automatically have C^1 trajectories.

1. INTRODUCTION

We study the incompressible Euler equations on a domain $\Omega := [0, 2\pi]^n$ in the absence of external forcing. These equations are classically written in terms of a divergence-free vector field u (i.e. $\nabla \cdot u = 0$) as follows:

$$(1) \quad \frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0$$

where p is a scalar potential representing internal pressure (as opposed to physical pressure at a boundary). The divergence-free condition reflects the incompressibility constraint.

In two and particularly in three dimensions, these equations are currently the subject of a great deal of study, see for example [5, 8]. As an illustration of the challenge posed by these equations we note that unlike the Navier-Stokes equations where global weak solutions have been known to exist since 1934 due to Leray, existence of global weak solutions of the Euler equations (on periodic domains) was not proved until 2011 by Wiedemann [12], following the work of DeLellis and Székelyhidi [7]. On the spatial domain \mathbb{R}^3 , more regular local solutions ($u \in C(0, T; H^s) \cap C^1(0, T; H^{s-2})$ with $s > 5/2$) have been known to exist since the 1970s due to Kato et al, see for example [9].

In the study of the Navier-Stokes equations, results such as those found in [10] motivate us to look at the classical equations of fluid mechanics from a more Lagrangian viewpoint. In that paper, Robinson and Sadowski show that if u is a suitable weak solution of the Navier-Stokes equations in 3D in the sense of Caffarelli, Kohn and Nirenberg [1], then almost every particle trajectory is unique and C^1 in time. The arguments there are based on the fact that almost all trajectories

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avoid the set of points (x, t) where singularities could develop using the fact that the set of such points has a box-counting dimension of at most $5/3$.

Constantin has found a form for the Euler equations that involves both the classical velocity field and the so called back-to-labels map A which is defined to be the inverse of the trajectory map X at each time t . More precisely, for an evolving vector field u defined on $\Omega \times [0, T]$, the trajectory map solves

$$(2) \quad \begin{cases} \frac{dX}{dt}(y, t) = u(X(y, t), t) \\ X(y, 0) = y \end{cases}$$

for each $y \in \Omega$. If u is divergence-free and sufficiently regular then X is well defined and $X(\cdot, t)$ is bijective for each t . In this case we can define the back-to-labels map A by setting

$$(3) \quad A(\cdot, t) = X^{-1}(\cdot, t),$$

where we consider X as a map $X(\cdot, t) : \Omega \rightarrow \Omega$ for each $t \in [0, T]$.

We shall review how the Euler equations can formally be rewritten in terms of A . Constantin showed that unique local-in-time solutions to this Eulerian-Lagrangian version of the equations exist in certain Hölder spaces. Here we show that a unique H^s solution exists locally in time on $\Omega = [0, 2\pi]^n$ for $s > \frac{n}{2} + 1$.

2. THE EULERIAN-LAGRANGIAN FORM OF THE EQUATIONS

The Eulerian-Lagrangian form of the Euler equations comprises the following system:

$$(4) \quad \partial_t A + (u \cdot \nabla) A = 0,$$

$$(5) \quad u = \mathbb{P}((\nabla A)^* v),$$

$$(6) \quad \partial_t v + (u \cdot \nabla) v = 0.$$

Given an initial divergence-free velocity u_0 for the classical equations, we choose initial conditions for the above system as follows:

$$(7) \quad A(x, 0) = x,$$

$$(8) \quad u(x, 0) = v(x, 0) = u_0(x).$$

We use the notation \mathbb{P} for the Leray projector onto the space of divergence-free functions. For a matrix M , M^* denotes the transposed matrix. The vector field v is called the *virtual velocity* and represents the initial velocity transported by the flow.

The following proposition encapsulates the derivation of (5) (sometimes called the Weber formula) which can be found in [2].

Proposition 1. *Let $n \geq 2$, consider $u \in C^1((0, T) \times \Omega)$, with $u(0) \in C^1(\Omega)$. If u is divergence-free and satisfies (1) for some p , with spatially periodic boundary conditions then $A \in C^1((0, T) \times \Omega; \Omega)$ and u satisfies (5) with virtual velocity $v(x, t) = u_0(A(x, t))$.*

Proof. From the regularity assumptions on u and periodicity of the domain we deduce that the trajectories $X(y, \cdot) \in C^2(0, T)$ and $\nabla X(y, \cdot) \in C^1(0, T)$ for all $y \in \Omega$, we also have $X, \frac{\partial X}{\partial t} \in C^1((0, T) \times \Omega)$. Moreover the divergence-free condition gives invertibility of ∇X so by the inverse function theorem A exists and is an element of $C^1((0, T) \times \Omega)$. We now have enough regularity to make the following calculations rigorous.

From (1) and (2) we obtain

$$\frac{\partial^2 X}{\partial t^2}(y, t) = -\nabla p(X(y, t), t),$$

which is of course just a Lagrangian interpretation of the Euler equations. Setting $\tilde{p}(y, t) = p(X(y, t), t)$ this becomes

$$\frac{\partial^2 X}{\partial t^2} = -((\nabla X)^*)^{-1} \nabla \tilde{p}(y, t).$$

Multiplying through by $(\nabla X)^*$ and changing the order of differentiation yields

$$(9) \quad \frac{\partial}{\partial t} \left[\frac{\partial X_j}{\partial t} \frac{\partial X_j}{\partial y_i} \right] = \frac{\partial}{\partial y_i} \left[-\tilde{p} + \frac{1}{2} \left| \frac{\partial X}{\partial t} \right|^2 \right]$$

for $i = 1, \dots, n$, where there is an implicit sum over $j = 1, \dots, n$ and X_j, y_i denote the components in \mathbb{R}^n of X, y respectively. Integrating (9) in time, multiplying the corresponding vector equation by $(\nabla A)^*$ and evaluating at $A(x, t)$ gives

$$(10) \quad u(x, t) = \frac{\partial X}{\partial t}(A(x, t), t) = (\nabla A)^* u_0(A(x, t)) - \nabla n$$

where

$$n(x, t) = \int_0^t \tilde{p}(A(x, t), s) - \frac{1}{2} \left| \frac{\partial X}{\partial t}(A(x, t), s) \right|^2 ds.$$

As gradients lie in the kernel of the Leray projector, applying \mathbb{P} to (10) shows that u satisfies (5) as required. Note that $v(x, t) = u_0(A(x, t))$ satisfies (6), hence solutions to the Euler equations indeed solve the Eulerian-Lagrangian form. \square

Since we are chiefly interested in the classical Euler equations we sketch a formal converse of Proposition 1. Let $D_t := \partial_t + (u \cdot \nabla)$ denote the material derivative. Suppose that u satisfies (5), we can write this as

$$(11) \quad u(x, t) = (\nabla A)^* v - \nabla n$$

for some real-valued n , with v satisfying (6). The key fact to note is the commutation relation

$$(12) \quad D_t \nabla f = \nabla D_t f - (\nabla u)^* \nabla f$$

for a suitably regular function f . This can easily be checked directly. Applying D_t to (11) and using (12) yields

$$\begin{aligned} D_t u &= (D_t \nabla A)^* v + (\nabla A)^* D_t v - D_t \nabla n \\ &= (\nabla D_t A)^* v - (\nabla u)^* (\nabla A)^* v - \nabla D_t n + (\nabla u)^* \nabla n \\ &= -(\nabla u)^* [(\nabla A)^* v - \nabla n] - \nabla D_t n \\ &= -(\nabla u)^* u - \nabla D_t n \\ &= -\nabla p \end{aligned}$$

where $p = \frac{1}{2} |u|^2 + D_t n$. This is nothing but (1), the classical Euler equations.

3. AN EXISTENCE AND UNIQUENESS THEOREM

For $r \geq 0$, let $H_{\text{per}}^r(\Omega)$ denote the space of periodic vector fields with components in $H^r(\Omega)$ (where H^0 is understood to be L^2). We will consider functions in the spaces $\Sigma_s(T)$ (usually denoted Σ_s) for $T \geq 0$ and $s \geq 0$ given by

$$\Sigma_s(T) := C(0, T; H_{\text{per}}^s(\Omega)).$$

We equip Σ_s with the natural norm:

$$\|u\|_{\Sigma_s} = \sup_{t \in [0, T]} \|u(t)\|_{H^s}.$$

The aim of the rest of this paper is to prove the following theorem.

Theorem 1. *If $n \geq 2$, $s > \frac{n}{2} + 1$ and*

$$u_0 \in H_{\text{per}}^s(\Omega)$$

then there exists $T > 0$, such that the system (4–6) with initial conditions (7) and (8) has a unique solution $A, u, v \in \Sigma_s(T)$. Moreover $A \in C^1([0, T] \times \Omega)$.

We will prove this by constructing a contracting iteration scheme using the equations (4–6). More precisely, given $u \in \Sigma_s(T)$ we find $A \in \Sigma_s \cap C^1([0, T] \times \Omega)$, $v \in \Sigma_s$, solutions of

$$\partial_t A + (u \cdot \nabla)A = 0, \quad A(0, x) = x$$

and

$$\partial_t v + (u \cdot \nabla)v = 0, \quad v(0, x) = u_0(x).$$

We then construct the next iterate of u , using

$$u' = \mathbb{P}[(\nabla A)^* v]$$

and show that $u \mapsto u'$ is a contraction on a certain subset of Σ_s .

In the case of Hölder spaces, Constantin constructed an iteration scheme that was instead a contraction with respect to A . This involves controlling differences between candidate virtual velocities (v_1 and v_2 , say) in terms of the difference between the respective back-to-labels maps (A_1 and A_2). This can be achieved, using the fact that $v_i = u_0(A_i)$ is a solution to (6). In the Hölder setting this is a natural way to proceed, however, relying on this *a posteriori* knowledge about the solution introduces an extra technicality when we work in Sobolev spaces. For this reason we will proceed as described above, relying only on *a priori* estimates. Following the proof, we shall see how the argument differs if the contraction is with respect to A , in particular we get an alternative proof under the additional assumption that $s \in \mathbb{Z}$.

We begin the proof of Theorem 1 by stating two inequalities concerning the advection term $(u \cdot \nabla)v$, using the notation $B(u, v) := (u \cdot \nabla)v$. Both of these results can be proved following the steps in [6, 11] (the only difference being that B here does not include a Leray projection).

Lemma 1. *For $s > \frac{n}{2}$ there exists $C > 0$ such that if $u \in H_{\text{per}}^s$ and $v \in H_{\text{per}}^{s+1}$ then $B(u, v) \in H_{\text{per}}^s$ and*

$$(13) \quad \|B(u, v)\|_{H^s} \leq C \|u\|_{H^s} \|v\|_{H^{s+1}}.$$

This is really just the fact that H^s is a Banach algebra. For the second lemma the assumption that u is divergence-free allows us to “save a derivative” by means of the identities

$$(B(u, (-\Delta)^{r/2}v), (-\Delta)^{r/2}v)_{L^2} = 0$$

for $r \in [0, s]$.

Lemma 2. *If $s > \frac{n}{2} + 1$ there exists $C > 0$ such that for $u \in H_{\text{per}}^s$, $v \in H_{\text{per}}^{s+1}$ with u divergence-free we have*

$$(14) \quad |(B(u, v), v)_{H^s}| \leq C \|u\|_{H^s} \|v\|_{H^s}^2.$$

We use the following shorthand for closed balls in Σ_s :

$$B_M = \overline{B_{\|\cdot\|_{\Sigma_s}}(0, M)},$$

i.e. B_M is the closed unit ball centred at the origin of radius $M > 0$ with respect to the norm $\|\cdot\|_{\Sigma_s}$. Where ambiguity could arise we write $B_M(T)$ for the closed ball in $\Sigma_s(T)$.

Lemma 3. *If $s > \frac{n}{2} + 1$ and $A, v \in \Sigma_s(T)$ then $\mathbb{P}[(\nabla A)^*v] \in \Sigma_s$ and there exists a constant $C_1 > 0$ (independent of A, v, t and T) such that for fixed t ,*

$$(15) \quad \|\mathbb{P}[(\nabla A)^*v]\|_{H^r} \leq C_1 (\|A\|_{H^s}) \|v\|_{H^r},$$

where $r = s$ or $r = s - 1$. Furthermore, there exists $C_2 > 0$ such that for any $M > 0$ and $T > 0$, the following bounds hold uniformly with respect to $t \in [0, T]$ for any $A_1, A_2, v_1, v_2 \in B_M(T)$:

$$(16) \quad \|\mathbb{P}[(\nabla A_1)^*v_1 - (\nabla A_2)^*v_2]\|_X \leq C_2 M (\|A_1 - A_2\|_X + \|v_1 - v_2\|_X),$$

where X is L^2 or H^{s-1} .

Proof. For continuity into H^{s-1} we use the fact that $H^{s-1}(\Omega)$ is a Banach algebra. More precisely, we see that

$$(17) \quad \begin{aligned} \|\mathbb{P}[(\nabla A_1)^*v_1 - (\nabla A_2)^*v_2]\|_{H^{s-1}} &\leq C \|A_1 - A_2\|_{H^s} \|v_1 + v_2\|_{H^{s-1}} \\ &\quad + C \|\nabla A_1 + \nabla A_2\|_{H^{s-1}} \|v_1 - v_2\|_{H^{s-1}}, \end{aligned}$$

where $C > 0$ is independent of the A_i and v_i . The key step in the proof of (15) when $r = s$ is that if $A, v \in C^2$ then for some $q \in H^s$,

$$\begin{aligned} \partial_{x_i} \mathbb{P}[(\nabla A)^*v] &= \partial_{x_i} (\partial_{x_j} A_k v_k) - \partial_{x_i} \partial_{x_j} q \\ &= \partial_{x_j} (\partial_{x_i} A_k v_k) - \partial_{x_i} A_k \partial_{x_j} v_k + \partial_{x_j} A_k \partial_{x_i} v_k - \partial_{x_i} \partial_{x_j} q \end{aligned}$$

where sums are taken implicitly over k . The left-hand side is already divergence-free so projecting again removes the gradient terms and yields

$$(18) \quad \partial_{x_i} \mathbb{P}[(\nabla A)^*v] = \mathbb{P}[(\nabla A)^* \partial_{x_i} v - (\nabla v)^* \partial_{x_i} A].$$

By continuity, this still holds if we only have $A, v \in H^s$. A calculation similar to (17) applied to (18) yields continuity with respect to the H^s norm as claimed.

The inequalities (15) for $r = s - 1$ and $r = s$ are obtained by taking the H^{s-1} norms of $\mathbb{P}[(\nabla A)^*v]$ and (18) respectively.

To prove (16), we again use the fact that \mathbb{P} removes gradients. Indeed for $f, g \in H^s$, we have

$$\mathbb{P}((\nabla f)^*g) = \mathbb{P}(\nabla(f \cdot g) - (\nabla g)^*f) = -\mathbb{P}((\nabla g)^*f).$$

Setting $f = A_1 - A_2$, $g = v_1 + v_2$, we see that the calculations in (17) can be modified to give the required result. Note that for the L^2 bound we use the fact that (17) holds if we replace H^s with L^∞ and H^{s-1} with L^2 . \square

The next lemma gives uniform bounds on the H^s norms of solutions to the transport equations (4) and (6). We will consider the following system:

$$(19) \quad \begin{cases} \partial_t f + (u \cdot \nabla) f = 0 \\ f(0) = f_0 \end{cases}$$

where $f : \Omega \rightarrow \mathbb{R}^n$ (or $f : \Omega \rightarrow \Omega$) and u is divergence free.

Lemma 4. *Let $s > \frac{n}{2} + 1$ and fix $f_0 \in H_{\text{per}}^s$. If $u \in \Sigma_s$ is divergence free then there exists a unique solution f to (19). Moreover $f \in \Sigma_s \cap C^1([0, T] \times \Omega)$ and there exists $C > 0$ (from Lemma 2) such that if $r, t \in [0, T]$ we have:*

$$(20) \quad \|f(t)\|_{H^s} \leq \|f(r)\|_{H^s} e^{C|t-r|\|u\|_{\Sigma_s}}.$$

Proof. By the method of characteristics we obtain a solution $f \in C^1([0, T] \times \Omega)$. The formal argument that follows motivates our consideration of the regularity of f . Taking the H^s product of (19) with f yields

$$\frac{1}{2} \frac{d}{dt} \|f\|_{H^s}^2 = -(B(u, f), f)_{H^s}.$$

By Lemma 2, there exists $C > 0$ such that for all $t \in [0, T]$,

$$(21) \quad \frac{d}{dt} \|f(t)\|_{H^s}^2 \leq C \|u(t)\|_{H^s} \|f(t)\|_{H^s}^2.$$

Now (20) follows from Gronwall's inequality. In the case $r > t$, this argument is applied to the time-reversed equation, that is, using the fact that for fixed r , $-f(r-t)$ is transported by $-u(r-t)$.

To properly justify this we proceed by a Galerkin method. For each $N \in \mathbb{N}$ we find a solution to the system

$$(22) \quad \begin{cases} \partial_t f_N + P_N B(u_N, f_N) = 0 \\ f_N(r) = P_N f(r), \end{cases}$$

on $[r, T]$, where P_N denotes truncation up to Fourier modes of order N (in space) and $u_N := P_N u$.

Using standard techniques, we obtain a weak solution $g \in L^\infty(r, T; H^s)$ such that $\partial_t g \in L^\infty(r, T; H^{s-1})$, hence $g \in C(0, T; H^{s-1})$. Using the divergence free property we obtain uniqueness of solutions $g \in L^2(r, T; H^1)$ with time derivative $\partial_t g \in L^2(r, T; L^2)$. Indeed, if g and \tilde{g} are two such solutions then

$$\int_r^t \frac{d}{ds} \|g - \tilde{g}\|_{L^2}^2 ds = 0.$$

Therefore $f = g$, i.e. this weak solution agrees with our C^1 classical solution on $[r, T]$.

We now prove (20). Since $f_N \rightarrow f$ in $L^\infty(r, T; H^{s-1})$, we may choose a dense countable subset $\{t_k\}_{k=1}^\infty \subset [r, T]$ such that $f_N(t_k) \rightarrow f(t_k)$ in H^{s-1} as $N \rightarrow \infty$ for each k . The formal argument above is valid on the truncated system, thus

$$(23) \quad \|f_N(t_k)\|_{H^s} \leq \|P_N f(r)\|_{H^s} e^{C|t_k-r|\|u\|_{\Sigma_s}}.$$

Hence, by a diagonalisation argument, we may assume that for all k , $f_N(t_k)$ converges weakly in H^s as $N \rightarrow \infty$. Moreover, by the choice of the points t_k and

uniqueness of weak limits, we must have $f_N(t_k) \rightharpoonup f(t_k)$ in H^s . Taking the lim sup of (23) with respect to $N \rightarrow \infty$ yields

$$\|f(t_k)\|_{H^s} \leq \|f(r)\|_{H^s} e^{C|t_k-r|\|u\|_{\Sigma_s}}.$$

Now (20) follows by the density of $\{t_k\}$ and the continuity of f into H^{s-1} .

For $t < r$ the required bounds are obtained in the same way from the time-reversed version of (22). By weak continuity, to see $f \in \Sigma_s$ it is enough that $\|f(t)\|_{H^s}$ is continuous, which is the case by (20) as required. \square

Lemma 5. *For $s > n/2 + 1$ fix $u_1, u_2 \in \Sigma_s$ and $f_0 \in H_{\text{per}}^s$. If f_1, f_2 are the solutions of (19) corresponding to u_1 and u_2 respectively, then there exists $C > 0$ depending only on s such that*

$$(24) \quad \|f_1(t) - f_2(t)\|_{L^2} \leq C\|u_1 - u_2\|_{\Sigma_0}\|f_1 + f_2\|_{\Sigma_s}t$$

for all $t \in [0, T]$.

Proof. Using the anti-symmetry of $(B(u_1 - u_2, \cdot), \cdot)_{L^2}$ we have, for $t \in [0, T]$,

$$\begin{aligned} \frac{d}{dt}\|f_1 - f_2\|_{L^2}^2 &\leq (B(u_1 - u_2, f_1 + f_2), f_1 - f_2)_{L^2} \\ &\leq C\|u_1 - u_2\|_{L^2}\|f_1 + f_2\|_{H^s}\|f_1 - f_2\|_{L^2} \\ &\leq C\|u_1 - u_2\|_{\Sigma_0}\|f_1 + f_2\|_{\Sigma_s}\|f_1 - f_2\|_{L^2} \end{aligned}$$

Where C depends on the embedding $H^{s-1} \hookrightarrow L^\infty$ and the constant from Lemma 2. Formally dividing by $\|f_1 - f_2\|_{L^2}$ and integrating the resulting inequality gives (24). Justifying this last step is straightforward. \square

We are now in a position to prove the main result.

Of Theorem 1. Fix $s > n/2 + 1$, let I denote the identity map on Ω and let C_1 be the constant in (15). Fix $M > C_1\|I\|_{H^s}\|u_0\|_{H^s}$ and $T > 0$ so that

$$C_1\|I\|_{H^s}\|u_0\|_{H^s}\exp(CMT) \leq M$$

where C is (in this case) the constant from Lemma 2. Let $u \in B_M(T)$ be a divergence free function and let A, v be solutions of (19), for the flow u and initial data I and u_0 respectively. Define $Su := \mathbb{P}[(\nabla A)^*v]$, then by Lemmas 3 and 4,

$$(25) \quad \|Su(t)\|_{H^s} \leq C_1\|I\|_{H^s}\|u_0\|_{H^s}e^{CMT} \leq M$$

for all $t \in [0, T]$. Hence $S : B_M(T) \rightarrow B_M(T)$.

We next show that S is a contraction on $B_M(T)$ in the L^2 norm if T is sufficiently small. Indeed, suppose $u_1, u_2 \in B_M(T)$ then

$$\begin{aligned} \|Su_1 - Su_2\|_{L^2} &\leq CM(\|A_1 - A_2\|_{L^2} + \|v_1 - v_2\|_{L^2}) \\ (26) \quad &\leq CMT\|u_1 - u_2\|_{\Sigma_0}(\|A_1 + A_2\|_{\Sigma_s} + \|v_1 + v_2\|_{\Sigma_s}) \\ &\leq CMTe^{CMT}\|u_1 - u_2\|_{\Sigma_0}(\|I\|_{H^s} + \|u_0\|_{H^s}), \end{aligned}$$

where C denotes various constants depending on constants from Lemmas 3, 4 and 5. Taking the supremum of (26) with respect to t and choosing $T > 0$ small enough, we see that S is a contraction in the required sense.

We conclude that S has a unique ‘‘accumulation point’’ u , in the closure of B_M with respect to $\|\cdot\|_{\Sigma_0}$. Since $B_M(T)$ is convex and closed in Σ_s it is weakly closed, hence $u \in B_M(T)$ is a fixed point of S . A fixed point of S , along with associated

back-to-labels map and virtual velocity, clearly give a solution to the Eulerian-Lagrangian formulation of the Euler equations with the required regularity. The contraction argument gives uniqueness in $B_M(T)$ and it remains to prove that we have uniqueness in $\Sigma_s(T)$.

Since S is a contraction on $B_M(T')$ for any $T' \in (0, T]$, we have by continuity of $\|u(t)\|_{H^s}$, that if u' , A' and v' also satisfy (4-6) with $u' \in \Sigma_s$, then $u(t) = u'(t)$ when $0 \leq t \leq \min(T, \inf\{r : \|u(r)\|_{H^s} = M\})$.

Now we know that for all $k \in \mathbb{N}$ there exists $T_k \leq T$ such that S is a contraction on $B_{M+1/k}(T_k)$ and we may assume $T_k \rightarrow T$ as $k \rightarrow \infty$. By the previous observation, this means that u is the unique solution in $\Sigma_s(T - \varepsilon)$ for all $\varepsilon > 0$, hence by continuity u is the unique solution in Σ_s as required. \square

4. AN ALTERNATIVE METHOD

Here we exhibit an alternative proof of existence and uniqueness for (4-6), which is based on contractions with respect to A rather than u . The extra technicality in this approach is contained in following lemma, which is proved in the appendix.

Lemma 6. *Let $s \in \mathbb{Z}$ with $s > \frac{n}{2} + 1$ and fix $f \in H^s(\Omega)$. If $g \in H^s(\Omega; \Omega)$ is volume preserving then $f \circ g \in H^s(\Omega)$ and*

$$(27) \quad \|f \circ g\|_{H^s} \leq C \|f\|_{H^s} \|g\|_{H^s}^s$$

This allows us to write a second proof of existence and uniqueness of solutions in Σ_s for $s > n/2 + 1$ in the case $s \in \mathbb{Z}$.

Fix $u_0 \in H^s$ and $M \geq \|I\|_{H^s}$ and suppose $A \in B_M(T)$ for some $T > 0$. Define u and v via $v = u_0 \circ A$ and $u = \mathbb{P}[(\nabla A)^* v]$. Construct A' , the iterate of A by solving

$$\partial_t A' + (u \cdot \nabla) A' = 0, \quad A'(x, 0) = x.$$

By Lemmas 3, 4 and 6, there exists $C > 0$ independent of T such that

$$\|A'\|_{\Sigma_s} \leq \|I\|_{H^s} \exp(CTM^{s+1}\|u_0\|_{H^s}).$$

Hence for T small enough, we may assume $A' \in B_M(T)$.

Now suppose that $A_1, A_2 \in B_M(T)$ and let A'_1, A'_2 be the respective iterates then

$$\|A'_1 - A'_2\|_{L^2} \leq CM^2 T \|A_1 - A_2\|_{L^2}$$

for all $t \in [0, T]$, by Lemmas 3 and 5. Here C depends on s and the Lipschitz constant of u_0 . It follows that, for small enough T , this iteration procedure is a contraction on $B_M(T)$ in the L^2 norm. Existence and uniqueness of solutions now follows using the same steps as in the previous method.

5. CONCLUSIONS

Constantin found that $C^{1,\mu}$ initial data gives rise to unique solutions with $C^{1,\mu}$ trajectories for a short time. In contrast, we have seen that for $s > n/2 + 1$, there exists a local solution which is continuous in time into H^s with trajectories in $C^1([0, T] \times \Omega)$.

This paper is partly to prepare the ground for a similar treatment of the Navier-Stokes equations. Once again it is Constantin [3, 4] who has put forward an Eulerian-Lagrangian form for the viscous case. In that formulation diffusive terms appear in the equations for the back-to-labels map and the virtual velocity and in the aforementioned papers some a priori information about that system and its

relationship to the classical Navier-Stokes equations are proved. We plan to consider a system for Navier-Stokes with a non-diffusive back to labels map and seek to prove a local existence result analogous to the one exhibited here.

APPENDIX A

In this appendix we prove Lemma 6, which gives bounds on the compositions of functions in H^s for $s \in \mathbb{Z}$ with $s > \frac{n}{2}$.

To begin with we consider $g_i \in H^s$ and $\beta_i \in [1, s]$ for $i = 1, \dots, \ell$. We call $p \in [1, \infty]$ *admissible* for $(\beta_i)_{1 \leq i \leq \ell}$ if there exists a constant $C > 0$ independent of $(g_i)_{1 \leq i \leq \ell}$ such that

$$(28) \quad \left\| \prod_{i=1}^{\ell} D^{\beta_i} g_i \right\|_{L^p} \leq C \prod_{i=1}^{\ell} \|g_i\|_{H^s}.$$

Of course p is admissible if there exist $q_1, \dots, q_\ell \in [1, \infty)$ so that $H^{s-|\beta_i|} \hookrightarrow L^{q_i}$ for each i and

$$\sum_{i=1}^{\ell} \frac{1}{q_i} = \frac{1}{p},$$

or $p = \infty$ and $q_i = \infty$ for all i . We may assume, without loss of generality that there are constants k_1 and k_2 with $0 \leq k_1 \leq k_2 \leq \ell$ such that

$$\begin{cases} s - |\beta_i| \in [0, n/2) \text{ for } 1 \leq i \leq k_1 \\ s - |\beta_i| = n/2 \text{ for } k_1 + 1 \leq i \leq k_2 \\ s - |\beta_i| > n/2 \text{ for } k_2 + 1 \leq i \leq \ell \end{cases}$$

So we have

$$\left\| \prod_{i=1}^{k_1} D^{\beta_i} g_i \right\|_{L^p} \leq C \prod_{i=1}^{k_1} \|g_i\|_{H^s}$$

for

$$\frac{1}{p} \in \left[\sum_{i=1}^{k_1} \frac{n - 2(s - |\beta_i|)}{2n}, \frac{k_1}{2} \right].$$

Moreover

$$\left\| \prod_{i=k_1+1}^{k_2} D^{\beta_i} g_i \right\|_{L^p} \leq C \prod_{i=k_1+1}^{k_2} \|g_i\|_{H^s}$$

for $p \in [2, \infty)$. Lastly,

$$\left\| \prod_{i=k_2+1}^{\ell} D^{\beta_i} g_i \right\|_{L^\infty} \leq C \prod_{i=k_2+1}^{\ell} \|g_i\|_{H^s}.$$

Combining these observations we see that p is admissible if

$$(29) \quad \frac{1}{p} \in \left(\sum_{i=1}^{k_1} \frac{n - 2(s - |\beta_i|)}{2n}, \frac{\ell}{2} \right].$$

or if $k_1 = k_2$ then p is still admissible if

$$(30) \quad \frac{1}{p} = \sum_{i=1}^{k_1} \frac{n - 2(s - |\beta_i|)}{2n},$$

furthermore $p = \infty$ is admissible if $k_1 = k_2 = 0$.

To prove the lemma we need to check that if $s > \frac{n}{2}$ and $\sum_{i=1}^{\ell} |\beta_i| \leq s$ then $p = 2$ is admissible for $(\beta_i)_{1 \leq i \leq \ell}$. Furthermore, we will need to show that if $s > n/2 + 1$ then there exists an admissible $p > \frac{n}{s-\ell}$ and that $p = \infty$ is admissible if $s = \ell$.

For the first claim, note that if $k_1 = 0$ or $k_1 = 1$ then $p = 2$ is clearly admissible. Otherwise, if $1 < k_1 \leq \ell$ and $s > n/2$, we have the following calculation:

$$(31) \quad \sum_{i=1}^{k_1} n - 2(s - |\beta_i|) \leq k_1 n - 2k_1 s + 2s = (k_1 - 1)(n - 2s) + n < n$$

so $p = 2$ is admissible. For the second claim, observe that if $s > n/2 + 1$ then

$$(32) \quad \sum_{i=1}^{k_1} n - 2(s - |\beta_i|) < 2 \sum_{i=1}^{k_1} |\beta_i| - 2k_1 \leq 2(s - k_1) - 2 \sum_{i=k_1+1}^{\ell} |\beta_i| \leq 2(s - \ell),$$

where the middle inequality uses the assumption that $\sum_{i=1}^{\ell} |\beta_i| \leq s$. Hence there exists an admissible value $p > \frac{n}{s-\ell}$, if $s - \ell > 0$. If $s = \ell$ then necessarily, $|\beta_i| = 1$ for $i = 1, \dots, \ell$ hence $p = \infty$ is admissible by (30).

Lemma 7. *Let $s \in \mathbb{Z}$ with $s > \frac{n}{2} + 1$ and fix $f \in H^s(\Omega)$. If $g \in H^s(\Omega; \Omega)$ is volume preserving then $f \circ g \in H^s(\Omega)$ and¹*

$$(33) \quad \|f \circ g\|_{H^s} \leq C \|f\|_{H^s} \|g\|_{H^s}^s$$

Proof. For each $k \in \mathbb{N}$, consider functions $f_k \in C^\infty(\Omega; \mathbb{R}^n)$ such that $f_k \rightarrow f$ in H^s . Similarly let $g_k \in C^\infty(\Omega; \Omega)$ be a sequence such that $g_k \rightarrow g$ in H^s . Without loss of generality we assume that $|\det \nabla g_k(x) - 1| < \frac{1}{k+1}$ holds uniformly in x .

Now by the chain and Leibniz rules, we see that for a multi-index γ with $|\gamma| \leq s$, $D^\gamma(f_k \circ g_k)$ is a (weighted) sum with summands of the form

$$(34) \quad ((D^\alpha f_k) \circ g_k) \prod_{i=1}^{\ell} D^{\beta_i} g_k^{r_i},$$

where $\ell = |\alpha|$ and $\sum_{i=1}^{\ell} |\beta_i| = |\gamma|$. Here g_k^i denotes the i th vector component of g_k . We seek to bound terms of the form (34) in L^2 using the preceding observations.

Since $D^\alpha f_k \in H^{s-\ell}$ and g_k is “almost volume preserving” it can be seen that $(D^\alpha f_k) \circ g_k \in L^q$ if

$$\frac{1}{q} \in \left(\frac{1}{2} - \frac{s-\ell}{n}, \frac{1}{2} \right]$$

with $s - \ell \in (0, n/2]$ or

$$\frac{1}{q} = \frac{1}{2} - \frac{s-\ell}{n}$$

when $s - \ell \in (0, n/2)$. Of course, if $s - \ell > n/2$ then $D^\alpha f \in L^\infty$.

To bound (34) in L^2 therefore, we need to check that there is an admissible p such that,

$$\frac{1}{p} \in \left[0, \frac{s-\ell}{n} \right).$$

and that $p = \infty$ is admissible if $s = \ell$. This follows from the claims we proved before the statement of the lemma.

¹To write the bound in this form, we use the fact that necessarily $\|g\|_{H^s} \geq 1$.

Now we see that

$$\|f_k \circ g_k\|_{H^s} \leq C \sqrt{1 + 1/k} \|f_k\|_{H^s} \|g_k\|_{H^s}^s$$

where C depends only on Sobolev constants and combinatorics. Since f_k and g_k converge we may assume that $f_k \circ g_k$ converges weakly in H^s . Thus the lemma is proved if we can show that $f_k \circ g_k \rightarrow f \circ g$ in L^2 for example. This is indeed the case:

$$\begin{aligned} \|f \circ g - f_k \circ g_k\|_{L^2} &\leq \|f \circ g - f \circ g_k\|_{L^2} + \|f \circ g_k - f_k \circ g_k\|_{L^2} \\ &\leq C \|g - g_k\|_{L^2} + \sqrt{1 + 1/k} \|f - f_k\|_{L^2}, \end{aligned}$$

where we make use of the fact that $f \in H^s$ is Lipschitz since $s > n/2 + 1$ and denote by C the Lipschitz constant of f . \square

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MATHEMATICS INSTITUTE, UNIVERSITY OF WARWICK, COVENTRY, CV4 7AL, UK
E-mail address: b.c.pooley@warwick.ac.uk

MATHEMATICS INSTITUTE, UNIVERSITY OF WARWICK, COVENTRY, CV4 7AL, UK
E-mail address: j.c.robinson@warwick.ac.uk